General Relativity $\boxed{\text{Week 12}}$

We stated last time:

Local well-posedness theorem for quasilinear wave eqns:

Theorem: Consider the initial value problem on $\mathbb{R}^{n+1}$:

$$\begin{cases} \partial_a \left(G^{a\beta}(\psi)\, \partial_\beta \psi \right) = N(\psi, \partial\psi) \\ \psi|_{t=0} = \psi_0, \quad \partial_t \psi|_{t=0} = \psi_1, \end{cases} \qquad \boxed{N(0,0) = 0}$$

Where $|G^{a\beta} - \eta^{a\beta}| < \frac{1}{10 \cdot n}$. Let also $k > 2n+2$

and set $M = \|\psi_0\|^2_{H^k} + \|\psi_1\|^2_{H^{k-1}}$. Then $\exists\, T = T(M) > 0$

and a unique solution ψ on $[0,T] \times \mathbb{R}^n$ such that

$$\|\psi\|^2_{L^\infty_t H^k_x} + \|\partial_t \psi\|^2_{L^\infty_t H^{k-1}_x} \leq C\, M, \qquad C\text{: Independent of } M$$

Moreover, ψ depends continuously on (ψ_0, ψ_1).

and $T(M) \to +\infty$ as $M \to 0$.

Proof:

- Existence: We will we a Picard iteration scheme.

Let us consider the sequence of functions $\psi^{(m)}$ on $\mathbb{R}^{n+1}$ defined recursively by solving the <u>linear</u> problems:

$$\begin{cases} \partial_\alpha \left(G^{\alpha\beta}(0)\, \partial_\beta \psi^{(0)} \right) = 0 \\ \psi^{(0)}|_{t=0} = \psi_0 , \ \partial_t \psi^{(0)}|_{t=0} = \psi_1 \end{cases}$$

and for $m \geq 1$:

$$\begin{cases} \partial_\alpha \left(G^{\alpha\beta}(\psi^{(m-1)})\, \partial_\beta \psi^{(m)} \right) = N(\psi^{(m-1)}, \partial\psi^{(m-1)}) \\ \psi^{(m)}|_{t=0} = \psi_0, \quad \partial_t \psi^{(m)}|_{t=0} = \psi_1 \end{cases}$$

(Linear eqns. unique solutions exist for all times).

Let $\mathcal{E}^{(k)}[f](\tau) = \sum_{|\alpha| \leq k} \int_{t=\tau} |\partial^\alpha f|^2 \, dx$

↑ also contains time derivatives.

~~We will show that:~~

Note: $\mathcal{E}^{(k)}[\psi^{(m)}](0) \sim M$.

We will show: If $T = T(M)$ is small enough:

① $$\begin{cases} \sup_{\tau \in [0,T]} \mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq C_0 M \\ \sup_{\tau \in [0,T]} \mathcal{E}^{(1)}[\psi^{(m)} - \psi^{(m-1)}] \leq \frac{1}{2} \sup_{\tau \in [0,T]} \mathcal{E}^{(1)}[\psi^{(m-1)} - \psi^{(m-2)}] \end{cases}$$

C_0: Uniform constant (say 100)

↑ Contracting sequence!

If ① holds: Then $\psi^{(m)}$ is a Cauchy sequence in $L^\infty_t H^1$ which is uniformly bounded in $L^\infty_t H^k$

$\Rightarrow$ by interpolation: It is a Cauchy sequence in $L^\infty_t H^s$ $\forall s < k$ (Interpolation inequality: $\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \cdot \|f\|_{H^{s_2}}^{\frac{s-s_1}{s_2-s_1}}$ if $s_1 < s < s_2$)

$\rightarrow$ Converges to limit ψ, Which satisfies

$$\sup_\tau \mathcal{E}^{(k)}[\psi](\tau) \leq C \cdot M.$$

Also since $k > 2n+2 > \frac{n}{2}+2$: By the Sobolev embedding, $\psi^{(m)}$ converges in C^2 to ψ, so ψ solves the equation.

In order to prove ①: We will argue inductively.

Assume it is true for all $m' \leq m-1$.

1^{st} order energy estimate:

Mult. $\partial_\alpha\left(G^{\alpha\beta}(\psi^{(m-1)})\,\partial_\beta \psi^{(m)}\right) = N(\psi^{(m-1)}, \partial\psi^{(m-1)})$ with $\partial_t \psi^{(m)}$ and integrate by parts:

$$\int_{t=\tau}\left(-G^{00}(\psi^{(m-1)})(\partial_0\psi^{(m)})^2 + G^{ij}(\psi^{(m-1)})\,\partial_i\psi^{(m)}\,\partial_j\psi^{(m)}\right)dx$$

$$-\int_{t=0}\cdots \qquad \text{Ⓐ}$$

$$= \int_{0\leq t\leq\tau}\Big[\tilde{G}(\psi^{(m-1)})\cdot\partial\psi^{(m-1)}\,\partial\psi^{(m)}\,\partial\psi^{(m)} + N(\psi^{(m-1)}, \partial\psi^{(m-1)})\cdot\partial\psi^{(m)}\Big]\,dx\,dt$$

Because $\left| G^{\alpha\beta} - \eta^{\alpha\beta} \right| < \frac{1}{10} \cdot \frac{1}{n}$

The terms in the left hand side are

$$\frac{2}{3}\, \mathcal{E}^{(1)}[\psi^{(m)}](\tau) \leqslant \quad \cdot \quad \leqslant \frac{3}{2}\, \mathcal{E}^{(1)}[\psi^{(m)}](\tau)$$

So:

$$\mathcal{E}^{(1)}[\psi^{(m)}](\tau) \leqslant 3\, \mathcal{E}[\psi^{(m)}](0) + C \underbrace{\int_{0 \leqslant t \leqslant \tau} |\tilde{G}(\psi^{(m-1)})| \cdot |\partial\psi^{(m-1)}| \cdot |\partial\psi^{(m)}|^2 + |N(\psi^{(m-1)}, \partial\psi^{(m-1)})| \cdot |\partial\psi^{(m)}| \, dx\,dt}_{B}$$

Inductively: $\|\psi^{(m-1)}\|_{L^\infty} + \|\partial\psi^{(m-1)}\|_{L^\infty} \underset{\substack{\text{Sobolev:} \\ k > \frac{n}{2}+1}}{\lesssim} \left(\mathcal{E}[\psi^{(m-1)}]\right)^{1/2} \leqslant C' M^{1/2}$

So: $\sup |\tilde{G}(\psi^{(m-1)})| \leqslant \sup\limits_{z \in [-C'M^{1/2}, C'M^{1/2}]} |\tilde{G}(z)| := C_1(M)$

and $\sup |N(\psi^{(m-1)}, \partial\psi^{(m-1)})| \leqslant C_2(M) \cdot M^{1/2}$

(Since $N(0,0) = 0$, I can write $N(x,y) = N_1(x,y) \cdot x + N_2(x,y) \cdot y$

So $B \leqslant C''(M) \int_{0 \leqslant t \leqslant \tau} M^{1/2} |\partial\psi^{(m)}|^2 + M^{1/2} |\partial\psi^{(m)}| \leqslant C''(M) \cdot \left(MT + \int_0^\tau \mathcal{E}[\psi^{(m)}]\, dt \right)$

Or $\mathcal{E}[\psi^{(m)}](\tau) \leq 3\,\mathcal{E}[\psi^{(m)}](0) + C''(M)\cdot M\cdot T$

$$+ C''(M)\cdot \int_0^{\tau} \mathcal{E}[\psi^{(m)}](s)\,ds$$

If $T = T(M)$ is small enough:

By Gronwall I get $\mathcal{E}[\psi^{(m)}](\tau) \leq$

$$e^{C''(M)\cdot T}\left(3\,\mathcal{E}[\psi^{(m)}](0) + C''M\cdot T\right)$$

$$\leq C_0 M$$

↑ the original constant!

I need to obtain the same bound for $\mathcal{E}^{(k)}[\psi^{(m)}](\tau)$

So: Higher order energy estimates.

Commute with ∂^a, $|a| \leq k-1$, repeat the same process:

$$\mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq 3\,\mathcal{E}^{(k)}[\psi^{(m)}](0)$$

$$+ C\int_0^{\tau} \tilde{F}\left(\partial^{\leq \frac{k}{2}+1}\psi^{(m-1)}\right)\cdot\left(|\partial^{\leq k}\psi^{(m-1)}|^2 + |\partial^{\leq k}\psi^{(m)}|^2\right)\,dx\,d\tau$$

↗ term with the fewest number of derivatives

Since $\|\partial^{\leq \frac{k}{2}+1} f\|_{L^\infty} \lesssim \|f\|_{H^k}$ $(k > 2n+2)$,

By using as before the inductive assumption & Gronwall:

If $T(M)$ is small enough, $\mathcal{E}^{(k)}[\psi^{(m)}](\tau) \leq C_0\, M$.

For the difference estimate:

Set $v^{(m)} = \psi^{(m)} - \psi^{(m-1)}$

Then $v^{(m)}$ satisfies a "linear" equation with coefficients depending on $\cancel{\psi}, \psi^{(m-1)}, \psi^{(m-2)}$:

$$G^{\alpha\beta}(\psi^{(m-1)})\, \partial_\alpha \partial_\beta v^{(m)} + A(\psi^{(m-2)}, \psi^{(m-1)}, \partial\psi^{(m-1)}, \partial^2 \dots)\, \partial v^{(m)}$$
$$+ B(\dots)\, v^{(m)}$$

$$= \tilde{A}(\dots)\, \partial v^{(m-1)} + \tilde{B}(\dots)\, v^{(m-1)}$$

To see this: Subtract the equations for $\psi^{(m)}$ and $\psi^{(m-1)}$ and for the difference of the non-linear terms we the "Taylor" formula:

$$F(\psi^2) - F(\psi^1) = \underbrace{\left(\int_0^1 F'(\lambda\psi^2 + (1-\lambda)\psi^1)\, d\lambda\right)}_{\tilde{F}(\psi^1, \psi^2)} (\psi^2 - \psi^1)$$

Applying the energy estimate for the equation for the difference:

$$\mathcal{E}[v^{(m)}](\tau) \leq 3 \cdot \underbrace{\mathcal{E}[v^{(m)}](0)}_{= 0 \text{ (same initial data)}} + C \| \partial^{\leq 2} \psi^{(m-2, m-1, m)} \|_{L^\infty} \int_0^\tau \mathcal{E}[v^{(m)}] + \mathcal{E}[v^{(m-1)}]$$

So by Gronwall: If $T(M)$ is small enough:

$$\sup_{\tau \in [0,T]} \mathcal{E}[v^{(m)}](\tau) \leq \frac{1}{100} \sup_{\tau \in [0,T]} \mathcal{E}[v^{(m-1)}](\tau).$$

Uniqueness / continuous dependence:

Apply the energy estimate to the difference of two solutions. $\square$

Remark:

- The same proof would work for quasilinear equations with principal symbol coefficient $G^{\alpha\beta}(\psi, \partial\psi)$.

- If $N(0,0) = 0$: $\psi = 0$ is a solution.

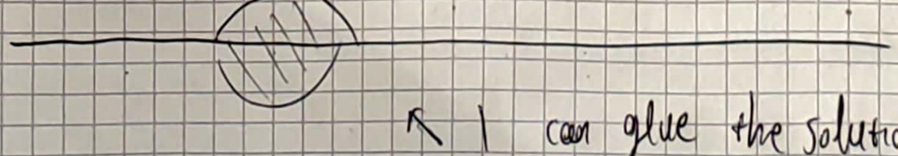
  Continuous dependence on initial data as $(\psi_0, \psi_1) \xrightarrow{H^k \times H^{k-1}} (0,0)$ means that:

  - $T \longrightarrow +\infty$
  - $\forall\, T_0$: on $[0, T_0] \times \mathbb{R}^n$ we have $\|\psi\|_{L^\infty_t H^k} + \|\partial_t \psi\|_{L^\infty H^{k-1}} \longrightarrow 0$

  (So: uniform over compact time domains).

- If I consider the well-posedness question in $H^k \times H^{k-1}$ for k "small": Might not be true - see exercises today.

For more general equations:



I can glue the solutions constructed in small regions